PLANE - PARALLEL MOTION APECTS REGARDING ISOCLINE CIRCLES

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Abstract: the paper studies isocline circles in plane-parallel motion. It has been found that a pair of such circles result, inflection and turning circles representing two borderline situations in which the pair of isocline circles merge into one single circle.

Key words: perforator, PETs, waste processing, construction and functioning

1. INTRODUCTION

In plane-parallel motion, throughout the movement of a rigid solid body, a plane (section) belonging to it overlays a fixed plane in space. Plane section (*S*) is considered, and we choose fixed reference system $O_1x_1y_1z_1$, so that $O_1x_1y_1$ plane would coincide with (*S*) section plane, and a mobile reference *Oxyz* system linked to the body, with *Oxy* plan in O_1x_1y fixed plane, and *Oz* and O_1z_1 axes being parallel (Fig. 1). The position of *O* point is determined by its position vector, and the rotation of the mobile benchmark, by θ angle between O_1x_1 and *Ox* axes. It results that the plane movement has three degrees of freedom.

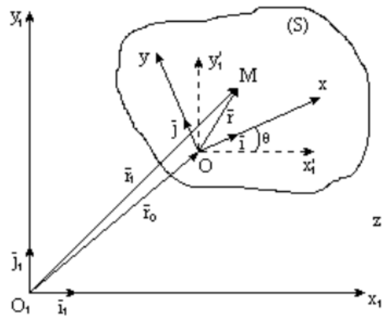

Fig. 1. *S* section in plane motion

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If the rigid body is subject to other links as well, the number of degrees of freedom will be reduced according to the links, to two or one degree of freedom.

The locus of the points of a rigid body in plane - parallel motion, which at a certain point have the same angle between speed and acceleration, forms the *isocline circles*, and in the case in which the speed and the acceleration are colinear vectors, there is a circle called *inflection circle*, and the locus of the points in which these two vectors are orthogonal, is also a circle called *tangent circle* (of escape).

2. GRAPHIC CONSTRUCTION OF ISOCLINE CIRCLES

In case of plane-parallel motion, there are situations in which the interest lies in specifying the locus of points in (S) section, in the motion plane for which at a given moment the angles between the speed and acceleration vectors of the same point have a given value α.

At the moment considered, the positions of I instantaneous rotation center of J accelerations pole and of φ angle are assumed to be known (given by the formula $tg\varphi = \frac{\varepsilon_I}{\omega_I^2}, \ \ \varphi \in \left[0, \frac{\pi}{2}\right)$).

Let a point M be of the locus sought (Fig. 2). For the beginning, we consider the case in which the acceleration vector is situated between the speed vector and the segment MJ and $\alpha < \pi/2$. From point M, IJ segment is seen in angle $\beta = |\pi/2 - \alpha - \varphi|$, so that if α and φ are the same for all M sought points, then β angle also stays unmodified at the moment considered. It follows that from M, IJ segment is seen in the same β angle. All M points that have this property, are found on the circle arc capable of β angle, and which rests on I and J points which are considered steady, at the moment considered. The locus is the circle arc (C) circumscribed to ΔMIJ triangle, except $IM'J$ arc, but also (C') circle arc, which is the symmetric of (C) arc with IJ segment.

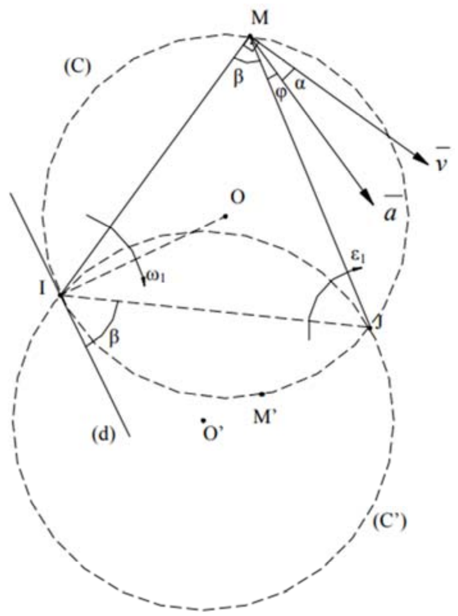

Fig. 2. Graphic construction of isocline circles

We mention that for M' position of M point (Fig. 2), α angle is greater than $\pi/2$. If we consider $\alpha > \pi/2$, then the locus for which we look is formed by the other two circle arcs, which themselves are symmetrical with the IJ segment, found “under” the IJ segment. When the $\overline{v}$ speed vector is included between the $\overline{a}$ acceleration vector and the *MJ* segment, then β angle stays constant but of $\beta = \pi/2 + \alpha - \varphi$ magnitude, the locus being still two circle arcs capable of the new β angle.

For the construction of a locus, we draw a (*d*) line in point *I*, so that the angle it forms with *IJ* would be β itself. Thus, line (*d*) becomes tangent to circle (*C*). The *O* center of the circle thus will be at the intersection of the perpendiculars in *I* on line (*d*) and in the middle of *IJ* on *IJ*.

In applications, two particular loci will be more important, called Bresse's circles. These are the circle of inflections, at which speed and acceleration of a point of the locus are colinear vectors, and the turning circle at which speed and acceleration of a point of the locus are perpendicular.

For the construction of these two circles, a line is drawn on both sides of the *IJ* segment, from point *I*, which will form angles φ and $\pi/2-\varphi$, respectively, with the segment *IJ* (Fig. 3). The perpendicular in *I* to *IJ* intersects the two lines in points *A,* and *B*, respectively. The *IA* diameter circle is the *inflection circle* since angle $\beta = \pi/2 - \varphi$, and the *BI* diameter circle is the *turning circle*, where $\beta = \varphi$.

If the angular speed is constant, it follows that the angular acceleration and thus φ angle are null.

The point *A* coincides with *I*, thus the inflection circle will have in this case *IJ* diameter, and the turning circle does not exist: there are no points that would have the speed and acceleration perpendicular to each other.

An important property of a point *M* on the inflection circle is the following: at a given moment, the speed being tangent to the trajectory, it results that the acceleration is tangent to the trajectory as well, so that the normal component of the acceleration is null: $a_n = v^2 / \rho_C$. But $v \neq 0$ it thus results $\rho_C = \infty$. It is the case of translation motion, a particular case of plane motion, but it is possible for point M to be a point in which the trajectory concavity changes, thus an inflection point, hence the name inflection circle.

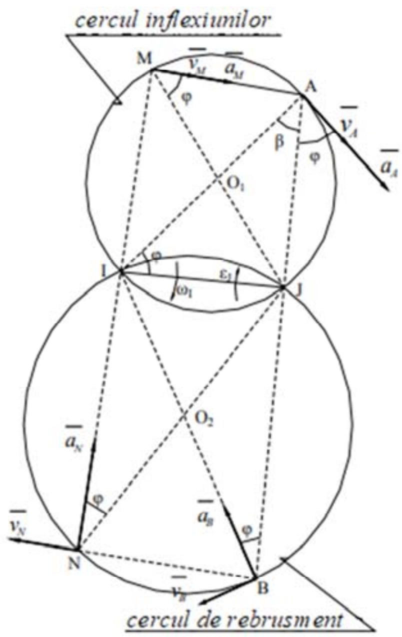

Fig. 3. Bresse's circles

In the case of a point *N* on the turning circle, acceleration is normal per speed, thus normal per trajectory, thus the tangent component of acceleration is null: $a_\tau = \dot{v} = 0$. At the moment considered, the point moves, thus the speed has an extreme value.

We note with β the angle made by vectors $\overline{AJ}$, $\overline{a}_A$ (see Figs. 4, 5). Then $tg\beta = \dfrac{|\varepsilon|}{\omega^2}$. Accelerations $\overline{a}_A$ of the points of the plate section are found at moment t on the same side of the radii JA, with which they make angle β. Agreeing that $\beta > 0$ if $\overline{a}_A$ is on the right of the JA segment, and $\beta < 0$, respectively, if $\overline{a}_A$ is on the left of the JA segment.

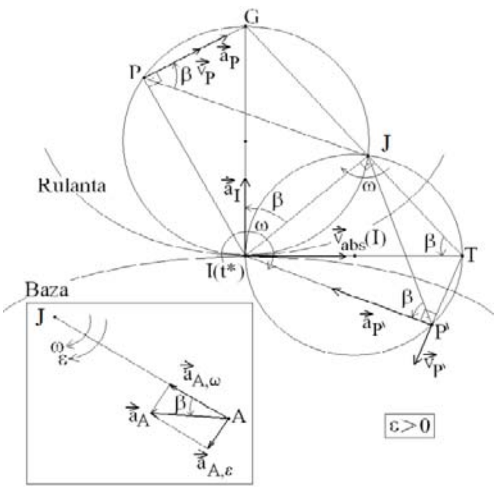

Fig. 4. Agreeing that $\beta > 0$ if $\overline{a}_A$ is on the right of the JA segment

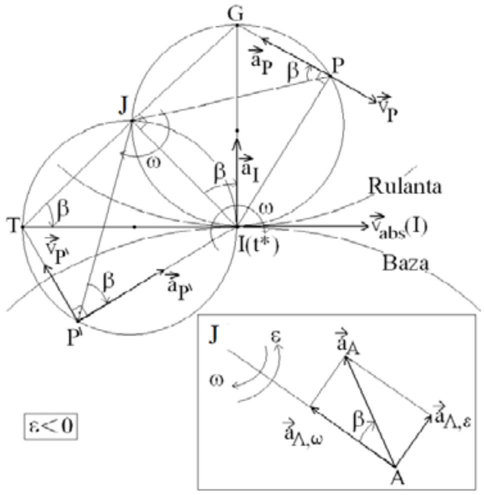

Fig. 5. Agreeing that $\beta < 0$ if $\overline{a}_A$ is on the left of the JA segment

3. ISOCLINE CIRCLES FROM KINEMATIC (ANALYTICAL) POINT OF VIEW

Cartesian equations will be further established, as well as the coordinates of the center and the radius of the isocline circles (as locus of the points of a rigid body in plane – parallel motion, which at a certain point have the same α angle between speed and acceleration), related to the xOy reference system joined with the rigid body.

It is noticed that a pair of such circles result, one corresponding to α angle, the other corresponding to its supplement (π-α). Bresses's circles represent two borderline situations when the pair of isocline circles merge into one single circle.

Considering that in some works the existence of isocline circles is demonstrated, by approaching the subject with the help of complex numbers at the most, without establishing their characteristics (the coordinate of the center and the radius), we propose to determine their cartesian equations, related to the reference system joined with the rigid body, the transition to the fixed reference system ($x_1O_1y_1$) being easily done.

For this, we initially need certain basic kinematic formulae.

Let's have a rigid body in plane-parallel motion, a fixed reference system $x_1O_1y_1$ and a reference system joined with the xOy rigid body (Fig. 6).

The motion of the rigid body is completely determined if the following functions are known:

$$x_{10} = x_{10}(t);\quad y_{10} = y_{10}(t);\quad \theta = \theta(t). \tag{1}$$

which represents the coordinates of the origin of the mobile system joined to the rigid body and the angle between the Ox_1' and Ox axes (see Fig. 6)

Transition from one system to another is done by the equations:

$$\begin{aligned} \overline{i_1} &= \overline{i}\,\cos\theta - \overline{j}\,\sin\theta; \\ \overline{j_1} &= \overline{i}\,\sin\theta - \overline{j}\,\cos\theta. \end{aligned} \tag{2}$$

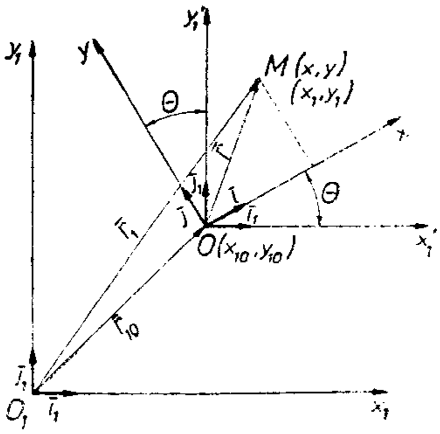

Fig. 6. Fixed and mobile system

$$x_1 = x_{10} + x\cos\theta - y\sin\theta;\quad y_1 = y_{10} + x\sin\theta - y\cos\theta. \tag{3}$$

where $\overline{i}$, $\overline{j}$, $\overline{i_1}$, $\overline{j_1}$ are the vectors of xOy, and $x_1O_1y_1$, system, respectively, and x, y, x_1, y_1 are the coordinates of a current point of the rigid body, in relation to the two reference systems.

The expressions of speed and acceleration of the same current point M of the rigid body, related to the two reference systems, are:

$$\begin{aligned} \overline{v_1} &= \dot{x}_1\overline{i_1} + \dot{y}\cdot\overline{j_1} = \\ &= \left(\dot{x}_{10} - \omega\cdot x\cdot\sin\theta - \omega\cdot y\cdot\cos\theta\right)\overline{i_1} + \\ &+ \left(\dot{y}_{10} + \omega\cdot x\cdot cos\,\theta - \omega\cdot y\cdot\sin\theta\right)\overline{j_1} \end{aligned} \tag{4}$$

$$\begin{aligned} \overline{a_1} &= \ddot{x}_1\cdot\overline{i_1} + \ddot{y}\cdot\overline{j_1} = \\ &= \left(\ddot{x}_{10} - \varepsilon\cdot x\cdot\sin\theta - \varepsilon\cdot y\cdot\cos\theta - \omega^2\cdot x\cdot cos\,\theta + \omega^2\cdot y\cdot\sin\theta\right)\overline{i_1} + \\ &+ \left(\ddot{y}_{10} + \varepsilon\cdot x\cdot cos\,\theta - \varepsilon\cdot y\cdot\sin\theta - \omega^2\cdot x\cdot\sin\theta - \omega^2\cdot y\cdot cos\,\theta\right)\overline{j_1} \end{aligned} \tag{5}$$

$$\overline{v} = \overline{v}_O + \overline{\omega}\times\overline{r} = \left(v_{Ox} - \omega\cdot y\right)\overline{i} + \left(v_{Oy} + \omega\cdot x\right)\overline{j} \tag{6}$$

$$\begin{aligned} \overline{a} &= \overline{a}_O + \overline{\varepsilon}\times\overline{r} + \overline{\omega}\times\left(\overline{\omega}\times\overline{r}\right) = \\ &= \left(a_{Ox} - \varepsilon\cdot y - \omega^2\cdot x\right)\overline{i} + \left(a_{Oy} + \varepsilon\cdot x + \omega^2\cdot y\right)\overline{j} \end{aligned} \tag{7}$$

where $\omega = \dot{\theta}$ and $\varepsilon = \ddot{\theta}$.

Assuming that functions (1) are known,

$$\begin{aligned} v_{Ox} &= \dot{x}_{10} \cos\theta + \dot{y}_{10} \sin\theta \\ v_{Oy} &= -\dot{x}_{10} \sin\theta + \dot{y}_{10} \cos\theta \end{aligned} \tag{8}$$

$$\begin{aligned} a_{Ox} &= \ddot{x}_{10} \cos\theta + \ddot{y}_{10} \sin\theta \\ a_{Oy} &= -\ddot{x}_{10} \sin\theta + \ddot{y}_{10} \cos\theta \end{aligned} \tag{9}$$

the terms of equations (6) and (7) can be written in the form:

$$\begin{aligned} v_x &= \dot{x}_{10} \cos\theta + \dot{y}_{10} \sin\theta - \omega \cdot y; \\ v_y &= -\dot{x}_{10} \sin\theta + \dot{y}_{10} \cos\theta + \omega \cdot x. \end{aligned} \tag{10}$$

$$\begin{aligned} a_x &= \ddot{x}_{10} \cos\theta + \ddot{y}_{10} \sin\theta - \varepsilon y - \omega^2 x \\ a_y &= -\ddot{x}_{10} \sin\theta + \ddot{y}_{10} \cos\theta + \varepsilon x - \omega^2 y. \end{aligned} \tag{11}$$

The angle between speed and acceleration of a point of the rigid body being α, and the condition being for it to be constant, we obtain:

$$\overline{v} \cdot \overline{a} = |\overline{v}| \cdot |\overline{a}| \cos\alpha \tag{12}$$

which leads to:

$$v_x a_x + v_y a_y = \sqrt{v_x^2 + v_y^2} \cdot \sqrt{a_x^2 + a_y^2} \cos\alpha \text{ sau} \left(v_x a_x + v_y a_y\right)^2 = \left(v_x^2 + v_y^2\right)\left(a_x^2 + a_y^2\right)\cos^2\alpha$$

respectively:

$$v_x^2 a_x^2 \left(1 - \cos^2\alpha\right) + 2 v_x a_x v_y a_y \left(\sin^2\alpha + \cos^2\alpha\right) + v_y^2 a_y^2 \left(1 - \cos^2\alpha\right) - \left(v_x^2 a_y^2 + v_y^2 a_x^2\right)\cos^2\alpha = 0$$

or:

$$\left(v_x a_x + v_y a_y\right)^2 \sin^2\alpha - \left(v_x a_y - v_y a_x\right)^2 \cos^2\alpha = 0 \tag{13}$$

Finally resulting:

$$\left(v_x a_x + v_y a_y\right)\sin\alpha - \left(v_x a_y - v_y a_x\right)\cos\alpha = 0 \tag{14}$$

$$\left(v_x a_x + v_y a_y\right)\sin\alpha + \left(v_x a_y - v_y a_x\right)\cos\alpha = 0 \tag{15}$$

If in the equations (14) and (15) we impose $\alpha = \pi / 2$, and $\alpha = 0$, respectively, we obtain formulae that can be in the form $\overline{v} \cdot \overline{a} = 0$, and $\overline{v} \times \overline{a} = 0$, respectively.

From these formulae, the cartesian equations of the two Bresse circles can be obtained, established already above, by substituting the speed and acceleration components from equations (6) and (7). It results that Bresse's circles are particular cases of isocline circles.

As it has been shown, the coordinates of the instantaneous rotation center *I* and of the acceleration center, identically verify the equations of Bresse's circles, thus, implicitly, all isocline circles have the same intersection points, namely the instantaneous center of rotation *I* and the acceleration center *J*.

By introducing the speed and acceleration components from equations (6) and (7) in equations (14) and (15), the equations of the two reunited isocline circles become:

$$\begin{aligned}
&\left(x^2+y^2\right)^2\left(\omega\varepsilon\sin\alpha\pm\omega^3\cos\alpha\right)+\\
&+\left[\left(a_{Oy}\omega+v_{Oy}\varepsilon-v_{Ox}\omega^2\right)\sin\alpha\mp\left(a_{Ox}\omega-v_{Ox}\varepsilon-v_{Oy}\omega^2\right)\cos\alpha\right]x-\\
&-\left[\left(a_{Ox}\omega+v_{Ox}\varepsilon+v_{Oy}\omega^2\right)\sin\alpha\pm\left(a_{Oy}\omega-v_{Oy}\varepsilon+v_{Ox}\omega^2\right)\cos\alpha\right]y+\\
&+\left[\left(v_{Ox}a_{Ox}+v_{Oy}a_{Oy}\right)\sin\alpha\pm\left(v_{Ox}a_{Oy}-v_{Oy}a_{Ox}\right)\cos\alpha\right]=0
\end{aligned} \tag{16}$$

Equation that can be put in the form:

$$\left(x-x_{c_{1,2}}\right)^2+\left(y-y_{c_{1,2}}\right)^2=R_{1,2}^2 \tag{17}$$

where:

$$x_{c_{1,2}}=-\frac{\left(a_{Oy}\omega+v_{Oy}\varepsilon-v_{Ox}\omega^2\right)\sin\alpha\mp\left(a_{Ox}\omega-v_{Ox}\varepsilon-v_{Oy}\omega^2\right)\cos\alpha}{2\left(\omega\varepsilon\sin\alpha\pm\omega^3\cos\alpha\right)} \tag{18}$$

$$y_{c_{1,2}}=\frac{\left(a_{Ox}\omega+v_{Ox}\varepsilon+v_{Oy}\omega^2\right)\sin\alpha\pm\left(a_{Oy}\omega-v_{Oy}\varepsilon+v_{Ox}\omega^2\right)\cos\alpha\alpha}{2\left(\omega\varepsilon\sin\alpha\pm\omega^3\cos\alpha\right)} \tag{19}$$

$$R_{1,2}^2=x_{c_{1,2}}^2+y_{c_{1,2}}^2-\frac{\left(v_{Ox}a_{Ox}+v_{Oy}a_{Oy}\right)\sin\alpha\pm\left(v_{Ox}a_{Oy}-v_{Oy}a_{Ox}\right)\cos\alpha}{\omega\varepsilon\sin\alpha\pm\omega^3\cos\alpha} \tag{20}$$

Index *1* corresponds to the sign above when in equations (18), (19) and (20) alternative signs occur, and defines the isocline circles given by equation (14), and index *2* corresponds to the other sign and defines the isocline circles given by equation (15).

It can be seen that index *2* isocline circle corresponds to the locus of the points of the rigid body, which at a given moment, have α as the angle between speed and acceleration, while index *1* isocline circle corresponds to the locus of the point of the rigid body, which at a given moment have (π - α). as the angle between speed and acceleration. We can thus name the two circles, corresponding to the angle formed between speed and acceleration, additional isocline circles.

Similarly, it can be shown that the angle formed between the radii of the two circles that transit through intersection points *J* (or *I*) is (π - 2α). Thus, except Bresse's circles, for which it has been shown that they are orthogonal, they are also orthogonal in the additional isocline circles corresponding to $\alpha = \pi/4$.

Introducing equations (8) and (9) in equations (18), (19) and (20), the equations of additional isocline circles can be written, in the cases in which the three parameters indicated by equation (1) that define plane-parallel motion are known as well.

4. CONCLUSIONS

The graphic construction of isocline circles and Bresse's circles (inflection circle and turning circle) has been given.

Cartesian equations of isocline circles (the locus of points joined with a rigid body in plane-parallel motion, which at a certain moment have the same angle α between speed and acceleration), related to the reference system joined with the rigid body has been determined. It has been established that they are isocline circle pairs, one corresponding to α angle between speed and acceleration, the other corresponding to its addition (π - α).

The two circles have as intersection points the instantaneous rotation center *I* and the acceleration center *J*.

Speed field and acceleration field in plane-parallel motion of the rigid body, at a given moment, can be associated with fascicles additional isocline circle pairs, all intersecting in the two points *I* and *J*, Bresse's circles also being included.The latter being a particular case, a limit situation, in which the additional isocline circle pairs merge in one single circle, since for the turning circle α=α/2 and π-α=π/2, and for the inflection circle $\alpha = 0$ or $\alpha = \pi$, and $\pi - \alpha = \pi$, and $\pi - \alpha = 0$, respectively.

Having the Cartesian equations, isocline circles equations can be easily written, being possible to be easily related to the fixed reference system as well.

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