SYNTHESIS OF FOUR-BAR LINKAGE USING DISPLACEMENT EQUATIONS

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Abstract: The paper presents the synthesis of four-bar mechanism using Freudenstein equation.

Keywords: synthesis of four-bar mechanism, Freudenstein equation.

1. INTRODUCTION

Consider a planar four-bar linkage O_AABO_B (fig. 1). This linkage is characterized by having four revolutes with parallel axes, the distances between successive axes being the parameters a_1, a_2, a_3, a_4. The synthesis of four-bar linkages, or mecans the determination of the four parameters that will yield an approximation to a desired function between the input (crank) and output (folower) angles.

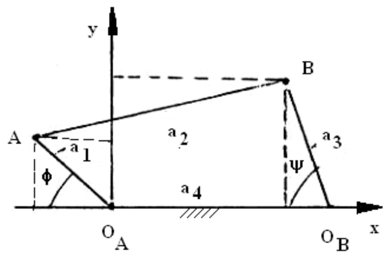

Fig. 1. Planar four-bar linkage; coordinates of A and B.

In this paper, algebraic methods for the synthesis of four-bar linkages as well as other planar mechanisms will be considered. Such methods of synthesis are based on displacement equations, i.e., equations relating the input and output variables of a mechanism in terms of its fixed parameters.

2. DISPLACEMENT EQUATION

The displacement equation of the four/bar linkage may be obtained bz considering a rectangular/coordinate system O_Axy (fig.1) with respect to which the coordinates of A and B may be written as follows:

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For A: $x_A = -a_1 \cos\phi$, $y_A = a_1 \sin\phi$

For B: $x_B = a_4 - a_3 \cos\psi$; $y_B = a_3 \sin\psi$

Since the distance AB is fixed and equal to a_2, application of Pythagoras theorem yields:

$$(x_B - x_A)^2 + (y_B - y_A)^2 = a_2^2$$
$$(a_4 - a_3 \cos\psi + a_1 \cos\phi)^2 + (a_3 \sin\psi - a_1 \sin\phi)^2 = a_2^2$$

After trigonometric simplifications this may be written:

$$A \sin\psi + B \cos\psi = C \tag{1}$$

where

$$A = \sin\phi, \qquad B = \frac{a_4}{a_1} + \cos\phi \qquad C = \frac{a_4}{a_3}\cos\phi + \frac{a_1^2 - a_2^2 + a_3^2 + a_4^2}{2a_1a_3}$$

Equation (1) may be solved for a desplacement analysis of the four-bar linkage; that is, ψ is found explicitly as a function of ϕ and the parameters a_1, a_2, a_3, a_4. Souch a solution is obtained by expressing $\sin\psi$ and $\cos\psi$ in terms of $\tan(\psi/2)$,

$$\sin\psi = \frac{2\tan\left(\frac{\psi}{2}\right)}{1 + \tan^2\left(\frac{\psi}{2}\right)}, \qquad \cos\psi = \frac{1 - \tan^2\left(\frac{\psi}{2}\right)}{1 + \tan^2\left(\frac{\psi}{2}\right)}$$

and substituting those values in Eq. (1) to get

$$2A\tan\left(\frac{\psi}{2}\right) + B\left(1 - \tan^2\frac{\psi}{2}\right) = C\left(1 + \tan^2\frac{\psi}{2}\right)$$

or

$$(B + C)\tan^2\frac{\psi}{2} - 2A\tan\frac{\psi}{2} - B + C = 0$$

from which

$$\tan\frac{\psi}{2} = \frac{A \pm \sqrt{A^2 + B^2 - C^2}}{B + C}.$$

For each value of ϕ the quantities A, B, C may be obtained and two distinct values of ψ found as:

Two distinct values of ψ found as:

$$\begin{aligned}\psi^{+} &= 2\arctan\frac{A+\sqrt{A^2+B^2-C^2}}{B+C} \\ \psi^{-} &= 2\arctan\frac{A-\sqrt{A^2+B^2-C^2}}{B+C}\end{aligned} \tag{2}$$

These two values correspond to the two ways in which a four-bar linkage may be closed (fig. 2.)

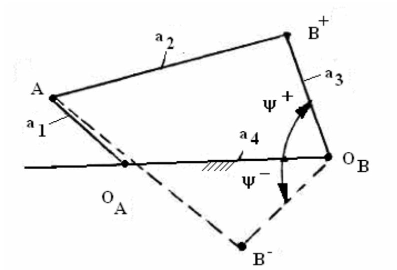

Fig. 2. Two solutions of the equation of the for-bar linkage

3. CRANK AND FOLLOWER SYNTHESIS: THREE ACCURACY POINTS

Consider the problem of designing a planar four-bar linkage such that to three given positions of the crank, defined by angles ϕ_1 , ϕ_2, and ϕ_3 , there correspond three prescribed positions of the follower, ψ_1 , ψ_2, and ψ_3. The solution consists in finding the proper values of a_1, a_2, a_3 and a_4 for three related pairs (ϕ_1, ψ_1), (ϕ_2, ψ_2), and (ϕ_3, ψ_3). The procedure is based on the Freudenstein displacement equation.

$$K_1\cos\phi - K_2\cos\psi + K_3 = \cos(\phi-\psi) \tag{3}$$

with: $K_1 = \dfrac{a_4}{a_3}$, $\quad K_2 = \dfrac{a_4}{a_1}$, $\quad K_3 = \dfrac{a_1^2 - a_2^2 + a_3^2 + a_4^2}{2a_1a_3}$

This equation was deduced from Eq.(1) by rearranging the terms. When written for three pairs of values, (ϕ_1, ψ_1), (ϕ_2, ψ_2), (ϕ_3, ψ_3), this equation yields a system of three equations linear with respect to K_1, K_2, K_3,

$$\begin{aligned}K_1\cos\phi_1 - K_2\cos\psi_1 + K_3 &= \cos(\phi_1-\psi_1)\\ K_1\cos\phi_2 - K_2\cos\psi_2 + K_3 &= \cos(\phi_2-\psi_2)\\ K_1\cos\phi_3 - K_2\cos\psi_3 + K_3 &= \cos(\phi_3-\psi_3)\end{aligned}$$

Tedious third-order determinants may be avoided by first subtracting the second and third equations from the first, thus eliminating K_3,

$$K_1(\cos\phi_1 - \cos\phi_2) - K_2(\cos\psi_1 - \cos\psi_2) = \cos(\phi_1 - \psi_1) - \cos(\phi_2 - \psi_2)$$
$$K_1(\cos\phi_1 - \cos\phi_3) - K_2(\cos\psi_1 - \cos\psi_3) = \cos(\phi_1 - \psi_1) - \cos(\phi_3 - \psi_3)$$

and solving the folloing resulting system of two equations with two unknowns,.

$$m_1K_1 - m_2K_2 = m_3$$
$$m_4K_1 - m_5K_2 = m_6$$

thus

$$K_1 = \frac{m_2m_6 - m_3m_5}{m_2m_4 - m_1m_5}; \quad K_2 = \frac{m_1m_6 - m_3m_4}{m_2m_4 - m_1m_5}$$

in which

$$\begin{cases} m_1 = \cos\phi_1 - \cos\phi_2 \\ m_2 = \cos\psi_1 - \cos\psi_2 \end{cases}; \begin{cases} m_3 = \cos(\phi_1 - \psi_1) - \cos(\phi_2 - \psi_2) \\ m_4 = \cos\phi_1 - \cos\phi_3 \end{cases};$$
$$\begin{cases} m_5 = \cos\psi_1 - \cos\psi_3 \\ m_6 = \cos(\phi_1 - \psi_1) - \cos(\phi_3 - \psi_3) \end{cases}$$

Substituting values of K_1 and K_2 into one of the three original equations yields K_3 as

$$K_3 = \cos(\phi_i - \psi_i) + K_1\cos\phi_i + K_2\cos\psi_i, \quad i = 1,2 \; or \; 3$$

With the values of K_1, K_2 and K_3, known, the parameters of the linkage may be found from the relations:

$$a_1 = \frac{a_4}{K_2}, \quad a_3 = \frac{a_4}{K_1}, \quad a_2 = \sqrt{a_1^2 + a_3^2 + a_4^2 - 2a_1a_3K_3}$$

The parameter a_4 may be given a positive but arbitrary value, usually taken as unity. This parameter merely determines the size of the linkage and has no effect on the angular relationships.

4. EXAMPLE: FOUR-BAR FUNCTION GENERATORS WITH THREE ACCURACY POINTS.

The design of four-bar function generators, is considered here as an application of the three-accuracy-point synthesis.

The function $y = log\,x$ is to be generated in the interval $1 \le x \le 2$ by means of a four-bar linkage O_AABO_B (fig. 3). The variables x and y are represented, respectively, by the crank and follower angles ϕ and ψ through the relations:

$$\frac{\phi-\phi_s}{\Delta\phi}=\frac{x-x_s}{\Delta x} \qquad \frac{\psi-\psi_s}{\Delta\psi}=\frac{y-y_s}{\Delta y}$$

Three accuracy points are taken in the interval $1 \le x \le 2$ with Chebyshev spacing (fig. 4) whence the corresponding values of the variables x and y are:

$x_1 = 1.0669$ $y_1 = 0.02812$
$x_2 = 1.5$ $y_2 = 0.17609$
$x_3 = 1.933$ $y_3 = 0.28623$

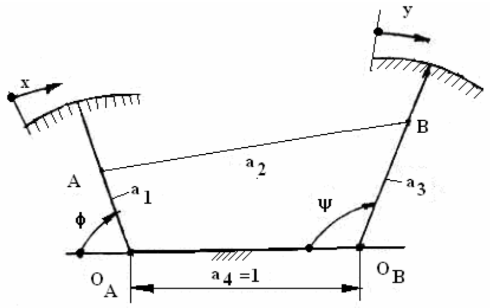

Fig. 3. Principle of four-bar-linkage function generator $y=f(x)$

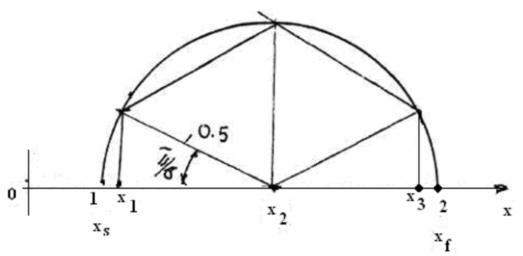

Fig. 4. Three accuracy points with Chebyshev spacing in the interval $1 \le x \le 2$

The ranges of variation of ϕ and ψ must be selected. They are chosen as $\Delta\phi = \Delta\psi = 60^0$. The rotations of the crank and follower from the position corresponding to the first accuracy point to the positions corresponding to the other two are, with the computation carried to $(1/10)^0$.

$$\phi_2-\phi_1=\frac{x_2-x_1}{x_f-x_s}\Delta\phi=25.986^0 \qquad \psi_2-\psi_1=\frac{y_2-y_1}{y_f-y_s}\Delta\psi=29.4927^0$$

$$\phi_3-\phi_1=\frac{x_3-x_1}{x_f-x_s}\Delta\phi=51.966^0 \qquad \psi_3-\psi_1=\frac{y_3-y_1}{y_f-y_s}\Delta\psi=51.44537^0$$

With the present method, the angles ϕ_1 and ψ_1, crank and follower positions corresponding to the first accuracy point, must also be selected at the start. Choosing $\phi_1= 45^0$ ($\phi_2= 30.986^0$, $\phi_3= 96.966^0$) and $\psi_1= 0^0$ ($\psi_2= 29.4927^0$, $\psi_3= 51.44537^0$) with a_4 =1.0, yielded a_1 =-1.031, a_2 =2.682, a_3 =2.310. These linkage proportions are favorable to force transmission, and the design may be considered as acceptable, if it is

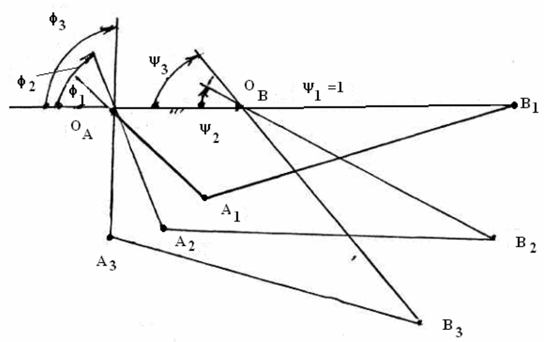

Fig. 5. One of the solutions of function generator linkage $y = \log x$; $1 \le x \le 2$, with three accuracy points.

recognized that it is a double rocker. The linkage, drawn in position 1, is shown in figure 5. The negative signs for a_1 are interpreted by considering O_AA as vector: the angles ϕ and ψ define O_AA and O_BB direction; the parameters a_1 and a_3 define their magnitudes and the sense in which they are to be laid off. A graphical check of this linkage for the three accuracy points shows that no large error is present. To determine the structural error accurately, an analysis must be carried aut by using Eqs.(2). The results of this analysis for values of ϕ in the interval $\phi_s \le \phi \le \phi_f$ at 6^0 intervals are summarized in Table 1.

Table 1. **Eror in log-function generator, three accuracy points**

x	ϕ, DEG	ψ, DEG	Log x	y_{mech}	y_{mech}-log x
1.0	40.986	-5.604757	0	-0.025	-0.025000
1.1	46.986	2.645517	0.041393	0.042	0.000607
1.2	52.986	10.163116	0.079181	0.080	0.000819
1.3	58.986	17.105870	0.113943	0.115	0.001057
1.4	64.986	23.520845	0.146128	0.146	-0.000128
1.5	70.986	29.492941	0.176091	0.176	-0.000091
1.6	76.986	35.079560	0.204120	0.204	-0.000120
1.7	82.986	40.327943	0.230449	0.230	-0.000449
1.8	88.986	45.275155	0.255273	0.255	-0.000273
1.9	94.986	49.955287	0.278754	0.279	0.000246
2.0	100.986	54.395243	0.301030	0.301	-0.000030

By taking $\psi = \psi^+$, the structural error, i.e., the difference between the values of y_{mech} given by the linkage and the corresponding values of log x, is shown in the last column. As expected, this structural error vanishes at the accuracy points. The maximum structural error, occurring at $x = 1.0$ is $e = -0.025$, or 8.3 percent of the range of variation of y.

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